Problem Set # 2

1. **Working with PDFs.** This problem should give you an idea of what the meaning and relevance of the higher moments of a probability distribution is.

   For a random variable U, consider a uniform PDF of the form

   $$f_a(V) = \begin{cases} 1/a & \text{if} \quad -a/2 \le V \le a/2 \\ 0 & \text{otherwise} \end{cases}$$

   and a piecewise uniform PDF of the form

   $$g_a(V) = \begin{cases} 1/2a & \text{if} \quad -a/2 \le V \le 0 \\ 3/2a & \text{if} \quad 0 \le V \le a/2 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Sketch f_a for $a = 1, a = 1/2,$ and $a = 0$. Sketch g_a for $a = 1$. What special distribution function does

   $$\lim_{a \to 0} f_a(V)$$

   correspond to?

   (b) Show that for f_a and g_a the normalization condition is satisfied.

   (c) Calculate $\langle U^n \rangle$ for $n = 1, ..., 4$ assuming that U is distributed according to f_a. From these, compute the variance, skewness, and kurtosis.

   (d) Repeat part (c) assuming that U is distributing according to g_a.

   (e) Using the previous results, discuss how these quantities describe the shape of the PDF and how the shape changes with respect to a. (Example: The variance describes how broad the distribution is. Increasing a leads to a broader distribution corresponding to a larger variance.)

2. **Isotropic Decaying Turbulence.** Using the isotropic decaying turbulence data from the previous homework:

   (a) Compute and plot the PDF of a velocity component. In isotropic turbulence, all three velocity components are statistically indistinguishable, so you can use all three components to get better statistics. Decribe the PDF by computing the mean, variance, skewness, and kurtosis. How do the values of the skewness and kurtosis compare to a Gaussian distribution?

   (b) From the PDF, can you estimate a characteristic length scale of the turbulence?

   (c) Discuss the fundamental properties of the Reynolds stress tensor using arguments about the symmetry of the flow. What can you say about the values on and off the diagonal?

   (d) Evaluate all six components of the Reynolds stress tensor by averaging over the entire field. Compare your results to what you expect.

3. **Joint Statistics.** Let U_1 be a standardized Gaussian random variable, and let U_2 be defined by $U_2 = U_1^2$.

   (a) Show that U_1 and U_2 are not independent.

   (b) Show that, although U_1 and U_2 are not independent, they are uncorrelated.

   Hint: While these statements can be proven with rather brief arguments, you can also explicity compute joint statistics with the joint PDF. The joint PDF is given by

$$f_{12}\left(V_1, V_2\right) = \delta\left(V_2 - V_1^2\right) f_1\left(V_1\right)$$